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The lateral migration of a spherical particle in two-dimensional shear flows

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The lateral migration of a solid spherical particle suspended in a fluid flowing between parallel vertical walls is investigated theoretically using a method developed by Cox & Brenner (1968). Buoyant and neutrally buoyant, freely rotating and non-rotating particles in the fluid flow are considered as is also the case of a sedimenting particle in a quiescent fluid. The results obtained are applied to the special cases of plane Poiseuille flow and of plane shear flow, these situations being investigated in detail.

1. Introduction

The lateral migration of a solid spherical particle suspended in a laminar tube flow has been demonstrated experimentally by Segré & Silberberg (1961, 1962). These observations, because of their importance to suspension rheology, have spawned a number of experimental studies concerning particle migration in tube flow (Oliver 1962; Eichhorn & Small 1964; Theodore 1964; Jeffrey & Pearson 1965; Denson, Christiansen & Salt 1966; Karnis, Goldsmith & Mason 1966*a*, *b*), in plane Poiseuille flow (Repetti & Leonard 1964, 1966; Tachibana 1973) and in Couette flow (Halow 1968; Halow & Wills 1970*a*, *b*). A number of theoretical studies (Rubinow & Keller 1961; Repetti & Leonard 1964, 1966; Saffman 1965) have also been undertaken. An extensive survey of existing experimental data and a critical analysis of the various theoretical attempts so far advanced to explain the phenomena has been presented by Brenner (1966), who concluded that none of these theories is capable of furnishing a satisfactory fundamental explanation of lateral migration in tubes because they take no account of either the presence of boundaries constraining the flow or of the variation of the shear rate across the tube.

Recently, the lateral migration of a spherical particle suspended in a flow bounded by solid boundaries has been studied theoretically in great generality by Cox & Brenner (1968). Using this general theory, Cox & Hsu (1975) succeeded in determining the migration velocity of a spherical particle, neutrally buoyant or not, suspended in a planar flow bounded by a single infinite plane wall. Their results are therefore expected to be valid for tube flow when the particle is situated

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not too far from the tube wall. The case of a freely rotating neutrally buoyant sphere, suspended in a planar flow bounded by two infinite plane walls, has been investigated by Ho & Leal (1974). Their analysis, based on the method of reflexions, is closely similar to the method of matched asymptotic expansions used by Cox & Brenner (1968). However, in the neighbourhood of the walls the results obtained by Ho & Leal (1974) do not seem to be in agreement with the asymptotic behaviour predicted by Cox & Hsu (1975). This discrepancy, which has not yet been explained, may be due to poor convergence of the numerical computation when the sphere is close to a wall.

In the present paper, the method developed by Cox & Brenner (1968) and Cox & Hsu (1975) is extended to the case in which the flow is in the vertical direction and is bounded by two vertical parallel plane walls. The migration velocity experienced by a spherical particle suspended in a Couette flow and in a plane Poiseuille flow is thus obtained. Cases of neutrally and non-neutrally buoyant particles are considered, the particles being either free to rotate or prevented from rotating by an external couple. The migration of a spherical particle sedimenting in a stagnant fluid bounded by two infinite plane walls is also considered. In this calculation, the results for the migration velocity obtained by Cox & Brenner (1968) are used. However, since these results are expressed in terms of volume integrals involving the Green's function for creeping flow in the presence of the given boundaries, one must first calculate this Green's function, which is accomplished by making a double Fourier transform of the flow field.

2. Lateral migration of a spherical particle

Consider a viscous fluid bounded by two parallel vertical plane walls W a distance l apart, either of these walls possibly moving in the vertical direction. A set of rectangular Cartesian co-ordinates (r_1', r_2', r_3') is chosen such that one of the walls is the plane $r_3' = 0$ and the other the plane $r_3' = l$. The undisturbed fluid flow velocity $\mathbf{U}'(\mathbf{r}')$ is assumed to vary only in the r_3' direction normal to the walls so that

$$\mathbf{U}'(\mathbf{r}') = (U_1'(r_3'), 0, 0). \tag{2.1}$$

Let a spherical particle of radius a be suspended in the fluid at a distance d from the wall $r_3' = 0$ (see figure 1). The dimensionless particle radius κ is then defined as

$$\kappa \equiv a/l, \tag{2.2}$$

and the particle Reynolds number Re as

$$Re = aV/\nu, \tag{2.3}$$

where V is the velocity with which the particle would move *upwards* in the r_1' direction as the result of gravity in an unbounded fluid at rest and ν is the kinematic viscosity of the fluid.

Using perturbation methods, Cox & Brenner (1968) have succeeded in obtaining a first-order expansion in Re of the Navier–Stokes and continuity equations subject to the appropriate boundary conditions on the spherical particle, walls

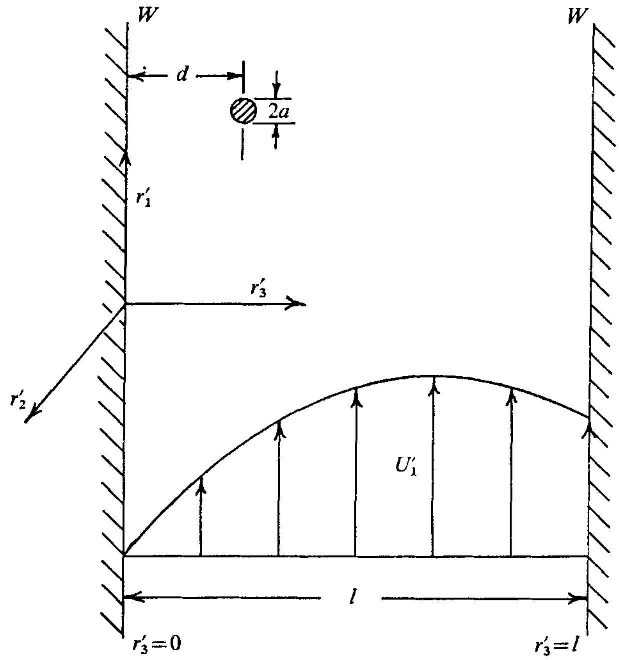


FIGURE 1. Spherical particle of radius a suspended in a fluid flowing between vertical parallel plates.

and infinity. They thus obtained the migration velocity of the particle towards or away from the walls. Their analysis assumes that the conditions

$$Re \ll 1, \quad \kappa \ll 1 \tag{2.4}$$

are satisfied so that a double expansion may be made in terms of the two parameters Re and κ. It is also required that the spherical particle should not be too close to either wall, i.e. it is assumed that

$$a/d \ll 1, \quad a/(l-d) \ll 1. \tag{2.5}$$

For expansions in terms of the Reynolds number Re, it may be seen that, in general, two regions of expansion exist, one being an inner region surrounding the particle in which viscous effects are dominant and the other being an outer region in which both viscosity and inertia are important. The expansion is thus singular and solutions can be obtained through matched asymptotic expansion techniques (see, for instance, Proudman & Pearson 1957; Rubinow & Keller 1961; Saffman 1965). However, in the present problem, Cox & Brenner (1968) have shown that if the inequality

$$Re/\kappa \ll 1 \tag{2.6}$$

is satisfied, that is if the walls are assumed to be located inside the inner region of expansion, one need consider only the inner expansion in order to calculate the first term in the expansion for the migration velocity. Furthermore, it was shown by Cox & Brenner that, if the parameter κ is small, this flow may be calculated by neglecting the size of the particle and assuming that the particle acts as a point force on the fluid. In this manner, values for the particle migration velocity were obtained in terms of integrals involving the Green's function for the solution of the creeping-flow equations. Throughout this calculation, the dimensionless variables used are based upon a characteristic velocity U_m of the undisturbed flow field and the distance l between the plane walls, so that the dimensionless position vector $\mathbf{r}$ and undisturbed flow velocity $\mathbf{U}(\mathbf{r})$ are defined as

$$\mathbf{r} = \mathbf{r}'/l, \quad \mathbf{U} = \mathbf{U}'/U_m. \tag{2.7}$$

The dimensional migration velocity v_l' of a spherical particle located at a distance d from the wall $r_3' = 0$ was obtained by Cox & Brenner for the following three cases:

(i) A particle in a fluid which is quiescent or undergoing a very slow motion with $|V/U_m| \gg 1$. The migration velocity for this case was found to be

$$v_l' = 6\pi(aV^2/\nu)\,h(\beta), \tag{2.8}$$

where $\beta = d/l$.

(ii) A non-neutrally buoyant spherical particle with the condition

$$\kappa^2 \ll |V/U_m| \ll 1$$

being satisfied. The migration velocity was found to be

$$v_l' = -6\pi(aV/\nu)\,U_m g(\beta). \tag{2.9}$$

(iii) A neutrally buoyant (or almost neutrally buoyant) spherical particle with $|V/U_m| \ll \kappa^2$,

(*a*) which is allowed to rotate, when

$$v_l' = \tfrac{10}{3}\pi\kappa^2(aU_m^2/\nu)f(\beta), \tag{2.10}$$

(*b*) which is not allowed to rotate, when

$$v_l' = \tfrac{4}{3}\pi\kappa^2(aU_m^2/\nu)\,p(\beta), \tag{2.11}$$

where $\beta = d/l$ as before.

The functions $h(\beta)$, $g(\beta)$, $f(\beta)$ and $p(\beta)$ appearing in (2.8)–(2.11) are defined explicitly by the following expressions, all integrals being over the entire fluid volume Γ (i.e. $0 < r_3 < 1$):

$$h(\beta) = \int\{\overline{V}_{i3}\,\partial\overline{V}_{i1}/\partial r_1\}d\mathbf{r}, \tag{2.12}$$

$$g(\beta) = \int\{[U_1(r_3) - U_1(r_3^*)]\,\overline{V}_{i3}\,\partial\overline{V}_{i1}/\partial r_1 + \overline{V}_{i1}\overline{V}_{13}\,\partial U_1(r_3)/\partial r_i\}d\mathbf{r}, \tag{2.13}$$

$$f(\beta) = [\partial U_1(r_3)/\partial r_3]_{r_3=r_3^*}[f_1(\beta) + f_2(\beta)], \tag{2.14}$$

$$p(\beta) = [\partial U_1(r_3)/\partial r_3]_{r_3=r_3^*}[4f_1(\beta) + f_2(\beta)], \tag{2.15}$$

$$f_1(\beta) = \int\left\{[U_1(r_3) - U_1(r_3^*)]\,\overline{V}_{i3}\frac{\partial^2\overline{V}_{i1}}{\partial r_3^*\,\partial r_1} + \frac{\partial U_1(r_3)}{\partial r_3}\,\overline{V}_{13}\frac{\partial\overline{V}_{31}}{\partial r_3^*}\right\}d\mathbf{r}, \tag{2.16}$$

$$f_2(\beta) = \int\left\{[U_1(r_3) - U_1(r_3^*)]\,\overline{V}_{i3}\frac{\partial^2\overline{V}_{i3}}{\partial r_1\,\partial r_1^*} + \frac{\partial U_1(r_3)}{\partial r_3}\,\overline{V}_{13}\frac{\partial\overline{V}_{33}}{\partial r_1^*}\right\}d\mathbf{r}, \tag{2.17}$$

where $\mathbf{r}^*$ is the dimensionless position vector of the centre of the sphere, so that $r_3^* = d/l = \beta$. The Cartesian tensor $\overline{V_{ij}}(\mathbf{r}, \mathbf{r}^*)$ is the Green's function for creeping

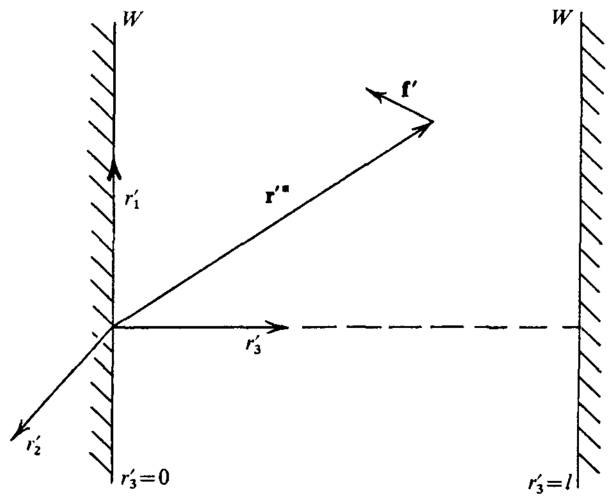


FIGURE 2. Point force of strength $\mathbf{f}'$ acting at position $\mathbf{r}'^*$.

flow between the planes W and thus satisfies

$$\bar{V}_{ij,kk} - \bar{P}_{j,i} + \delta_{ij}\delta(\mathbf{r}-\mathbf{r}^*) = 0, \quad \bar{V}_{ij,i} = 0, \tag{2.18}$$

with $\bar{V}_{ij} = 0$ on W; δ_{ij} is the Kronecker delta and $\delta(\mathbf{r}-\mathbf{r}^*)$ is the three-dimensional Dirac delta function. Thus $\bar{V}_{ij}$ physically represents the ith component of the creeping-flow velocity field at $\mathbf{r}$ due to a unit point force acting on the fluid at $\mathbf{r}^*$ in the jth direction.

3. Flow produced by a point force acting between two plane walls

In order to evaluate the expressions given in §2 for the migration velocity, we first calculate the Green's function $\bar{V}_{ij}$. Thus a single isolated point force is assumed to act at an arbitrary point within the fluid bounded by the two plane walls W. The strength of this point force is taken as $\mathbf{f}'$ and is assumed to act at a point $\mathbf{r}'^* = (r_1'^*, r_2'^*, r_3'^*)$ relative to the (r_1', r_2', r_3') rectangular Cartesian co-ordinate system (see figure 2).

The flow velocity $\mathbf{u}'$ and pressure p' produced by this point force $\mathbf{f}'$ are assumed to satisfy the creeping-flow equations

$$\mu\nabla'^2\mathbf{u}' - \nabla' p' + \mathbf{f}'\delta(\mathbf{r}'-\mathbf{r}'^*) = 0, \tag{3.1}$$

$$\nabla'.\mathbf{u}' = 0, \tag{3.2}$$

with the boundary condition

$$\mathbf{u}' = 0 \quad \text{on} \quad W, \tag{3.3}$$

where μ is the viscosity of the fluid.

Letting V and l be the characteristic velocity and length respectively and introducing the dimensionless variables

$$\mathbf{r} = \mathbf{r}'/l, \quad \mathbf{u} = \mathbf{u}'/V, \quad \nabla = l\nabla', \quad p = p'l/\mu V, \quad \mathbf{f} = \mathbf{f}'/\mu l V, \tag{3.4}$$

we can write (3.1)–(3.3) in dimensionless form as

$$\nabla^2\mathbf{u} - \nabla p + \mathbf{f}\delta(\mathbf{r}-\mathbf{r}^*) = 0, \tag{3.5}$$

$$\nabla . \mathbf{u} = 0, \tag{3.6}$$

$$\mathbf{u} = 0 \quad \text{on} \quad W. \tag{3.7}$$

Define $\mathbf{\Gamma}$ and Π as the two-dimensional Fourier transforms of the velocity $\mathbf{u}$ and pressure p respectively, so that

$$\mathbf{\Gamma}(k_1, k_2, r_3) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{u}(\mathbf{r}) \exp[-i(k_1 r_1 + k_2 r_2)]\, dr_1\, dr_2, \tag{3.8a}$$

$$\Pi(k_1, k_2, r_3) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\mathbf{r}) \exp[-i(k_1 r_1 + k_2 r_2)]\, dr_1\, dr_2, \tag{3.8b}$$

$\mathbf{u}$ and p then being given by the inverse Fourier transforms

$$\mathbf{u}(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{\Gamma}(k_1, k_2, r_3) \exp[i(k_1 r_1 + k_2 r_2)]\, dk_1\, dk_2, \tag{3.9a}$$

$$p(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi(k_1, k_2, r_3) \exp[i(k_1 r_1 + k_2 r_2)]\, dk_1\, dk_2. \tag{3.9b}$$

By taking the Fourier transforms of (3.5)–(3.7), it is seen that $\mathbf{\Gamma}$ and Π satisfy the relations

$$\left[-k_1^2 - k_2^2 + \frac{\partial^2}{\partial r_3^2}\right]\begin{pmatrix}\Gamma_1\\ \Gamma_2\\ \Gamma_3\end{pmatrix} - \begin{pmatrix} ik_1 \\ ik_2 \\ \dfrac{\partial}{\partial r_3}\end{pmatrix}\Pi + \frac{1}{4\pi^2}\begin{pmatrix} f_1\\ f_2\\ f_3\end{pmatrix}\delta(r_3 - r_3^*)\exp[-i(k_1 r_1^* + k_2 r_2^*)] = 0, \tag{3.10}$$

$$i(k_1\Gamma_1 + k_2\Gamma_2) + \partial\Gamma_3/\partial r_3 = 0, \tag{3.11}$$

subject to the boundary conditions

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = 0 \quad \text{at} \quad r_3 = 0, \tag{3.12a}$$

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = 0 \quad \text{at} \quad r_3 = 1. \tag{3.12b}$$

Multiplying the first component of (3.10) by ik_1, the second component by ik_2, adding the resulting equations and using the equation of continuity (3.11), one obtains the value of Π as

$$\Pi = \frac{1}{q^2}\frac{\partial^3\Gamma_3}{\partial r_3^3} - \frac{\partial\Gamma_3}{\partial r_3} - \frac{n}{4\pi^2 q}\delta(r_3 - r_3^*)\, e^{-\alpha}, \tag{3.13}$$

where $\quad q^2 = k_1^2 + k_2^2, \quad n = i(k_1 f_1 + k_2 f_2)/q \quad$ and $\quad \alpha = i(k_1 r_1^* + k_2 r_2^*)$.

Differentiating (3.13) with respect to r_3, substituting the resulting expression into the third component of (3.10) and integrating the resulting differential equation yields the value of Γ_3 as

$$\begin{aligned}\Gamma_3 = {} & (A + Br_3)\exp[-qr_3] + (C + Dr_3)\exp[qr_3] \\ & + \tfrac{1}{16}\pi^{-2}\{f_3(q^{-1} + |r_3 - r_3^*|) - n(r_3 - r_3^*)\}\exp[-q|r_3 - r_3^*| - \alpha],\end{aligned} \tag{3.14}$$

where A, B, C and D are constants of integration which have to be evaluated from the boundary conditions. Differentiating (3.14) with respect to r_3 once and thrice and substituting the resulting expressions into (3.13), one obtains the value of Π as

$$\Pi = 2B\exp[-qr_3]+2D\exp[qr_3] + \tfrac{1}{8}\pi^{-2}\{f_3|r_3-r_3^*|/(r_3-r_3^*)-n\}\exp[-q|r_3-r_3^*|-\alpha]. \quad (3.15)$$

By substituting this value of Π into the first and second components of (3.10) and integrating the resulting differential equations, one obtains the values of Γ_1 and Γ_2 as

$$\Gamma_1 = (G-ik_1q^{-1}Br_3)\exp[-qr_3]+(H+ik_1q^{-1}Dr_3)\exp[qr_3]+\tfrac{1}{16}\pi^{-2}\{2f_1q^{-1} + ik_1nq^{-2}(1+q|r_3-r_3^*|)-ik_1q^{-1}f_3(r_3-r_3^*)\}\exp[-q|r_3-r_3^*|-\alpha], \quad (3.16)$$

$$\Gamma_2 = (E-ik_2q^{-1}Br_3)\exp[-qr_3]+(F+ik_2q^{-1}Dr_3)\exp[qr_3]+\tfrac{1}{16}\pi^{-2}\{2f_2q^{-1} + ik_2nq^{-2}(1+q|r_3-r_3^*|)-ik_2q^{-1}f_3(r_3-r_3^*)\}\exp[-q|r_3-r_3^*|-\alpha], \quad (3.17)$$

where E, F, G and H are constants of integration.

By substituting (3.14), (3.16) and (3.17) into the boundary conditions (3.12*a*), one obtains the following expressions:

$$A+C = -\tfrac{1}{16}\pi^{-2}\{f_3q^{-1}(1+qr_3^*)+nr_3^*\}\exp[-qr_3^*-\alpha], \quad (3.18)$$

$$G+H = -\tfrac{1}{16}\pi^{-2}\{2f_1q^{-1}+ik_1nq^{-2}(1+qr_3^*)+ik_1q^{-1}f_3r_3^*\}\exp[-qr_3^*-\alpha], \quad (3.19)$$

$$E+F = -\tfrac{1}{16}\pi^{-2}\{2f_2q^{-1}+ik_2nq^{-2}(1+qr_3^*)+ik_2q^{-1}f_3r_3^*\}\exp[-qr_3^*-\alpha]. \quad (3.20)$$

Similarly from boundary conditions (3.12*b*), it is seen that

$$(A+B)\exp[-q]+(C+D)\exp[q] = -\tfrac{1}{16}\pi^{-2}\{f_3q^{-1}(1+q(1-r_3^*))-n(1-r_3^*)\} \times\exp[-q(1-r_3^*)-\alpha], \quad (3.21)$$

$$(E-ik_2q^{-1}B)\exp[-q]+(F+ik_2q^{-1}D)\exp[q] = -\tfrac{1}{16}\pi^{-2}[2f_2q^{-1}+ik_2nq^{-2}\{1+q(1-r_3^*)\}-ik_2q^{-1}f_3(1-r_3^*)]\exp[-q(1-r^*)-\alpha], \quad (3.22)$$

$$(G-ik_1q^{-1}B)\exp[-q]+(H+ik_1q^{-1}D)\exp[q] = -\tfrac{1}{16}\pi^{-2} \times[2f_1q^{-1}+ik_1nq^{-2}\{1+q(1-r_3^*)\}-ik_1q^{-1}f_3(1-r_3^*)]\exp[-q(1-r_3^*)-\alpha]. \quad (3.23)$$

Furthermore by substituting the values of Γ_3, Γ_2 and Γ_1 as given by (3.14), (3.17) and (3.16) into the continuity equation (3.11), one obtains the following additional relations between the constants $A, \ldots, H$:

$$ik_1G+ik_2E+B = Aq \quad (r_3-r_3^*) > 0, \quad (3.24)$$

$$ik_1H+ik_2F+D = -Cq \quad (r_3-r_3^*) < 0. \quad (3.25)$$

Equations (3.18)–(3.25) constitute 8 equations in the 8 unknowns $A, \ldots, H$

which can be solved to give

$$A = S^{-1}\sum_{i=1}^{4} m_i a_i, \quad B = S^{-1}\sum_{i=1}^{4} m_i b_i, \tag{3.26), (3.27}$$

$$C = -\tfrac{1}{16}\pi^{-2}m_1 - A, \quad D = S^{-1}\sum_{i=1}^{4} m_i d_i, \tag{3.28), (3.29}$$

$$E = -(n_1 e^q - n_2)/a_4, \quad F = (n_1 e^{-q} - n_2)/a_4, \tag{3.30), (3.31}$$

$$G = -(n_3 e^q - n_4)/a_4, \quad H = (n_3 e^{-q} - n_4)/a_4, \tag{3.32), (3.33}$$

where $a_1 = e^{2q} - 2q^2 + 2q - 1, \quad a_2 = -2q, \quad a_3 = -(1+q)e^q + (1-q)e^{-q},$

$$a_4 = -e^q + e^{-q}, \tag{3.34}$$

$$b_1 = q(e^{2q} + 2q - 1), \quad b_2 = 1 + 2q - e^{2q}, \quad b_3 = -q(e^q - e^{-q} + 2q e^q),$$

$$b_4 = e^q - e^{-q} - 2q e^q, \tag{3.35}$$

$$d_1 = q(1 + 2q - e^{-2q}), \quad d_2 = 1 - 2q - e^{-2q}, \quad d_3 = -q(e^q - e^{-q} + 2q e^{-q}),$$

$$d_4 = -e^q + e^{-q} + 2q e^{-q}, \tag{3.36}$$

$$m_1 = [nr_3^* + f_3(q^{-1} + r_3^*)]\exp[-qr_3^* - \alpha], \tag{3.37a}$$

$$m_2 = [n(1 - qr_3^*) - f_3 r_3^* q]\exp[-qr_3^* - \alpha], \tag{3.37b}$$

$$m_3 = [n(r_3^* - 1) + f_3(q^{-1} + 1 - r_3^*)]\exp[-q(1 - r_3^*) - \alpha], \tag{3.37c}$$

$$m_4 = [n\{1 - q(1 - r_3^*)\} + f_3 q(1 - r_3^*)]\exp[-q(1 - r_3^*) - \alpha], \tag{3.37d}$$

$$n_1 = -\tfrac{1}{16}\pi^{-2}[2f_2 + ik_2\{nq^{-1}(1 + qr_3^*) + f_3 r_3^*\}]\,q^{-1}\exp[-qr_3^* - \alpha], \tag{3.38a}$$

$$n_2 = ik_2(B e^{-q} - D e^q)/q - \tfrac{1}{16}\pi^{-2}[2f_2 + ik_2\{n(q^{-1} + 1 - r_3^*) - f_3(1 - r_3^*)\}]\,q^{-1}\exp[-q(1 - r_3^*) - \alpha], \tag{3.38b}$$

$$n_3 = -\tfrac{1}{16}\pi^{-2}[2f_1 + ik_1\{n(q^{-1} + r_3^*) + f_3 r_3^*\}]\,q^{-1}\exp[-qr_3^* - \alpha], \tag{3.38c}$$

$$n_4 = ik_1(B e^{-q} - D e^q)/q - \tfrac{1}{16}\pi^{-2}\{2f_1 + ik_1[n(q^{-1} + 1 - r_3^*) - f_3(1 - r_3^*)]\}\,q^{-1}\exp[-q(1 - r_3^*) - \alpha], \tag{3.38d}$$

$$S = -16\pi^2(e^{2q} + e^{-2q} - 4q^2 - 2). \tag{3.39}$$

The Fourier representation (Γ_1, Γ_2, Γ_3) of the flow field (u_1, u_2, u_3) produced by a point force (f_1, f_2, f_3) acting at an arbitrary point (r_1^*, r_2^*, r_3^*) within a stagnant fluid bounded by two rigid solid plane walls is thus given by (3.16), (3.17) and (3.14) with the coefficients $A, \ldots, H$ given by (3.26)–(3.39). The explicit forms for the expressions Γ_1, Γ_2 and Γ_3 obtained by substituting the coefficients $A, \ldots, H$ have not been written here because the resulting expressions are too lengthy. However, these expressions will be used in the analysis which follows.

4. The Green's function

In this section, the Green's function for creeping flow bounded by two rigid parallel plane walls W is considered. The flow field $\mathbf{u}(\mathbf{r})$ considered in the previous section, satisfying the creeping-flow equations with a point force $\mathbf{f}$ at $\mathbf{r} = \mathbf{r}^*$ and the no-slip condition on walls W, must depend linearly on $\mathbf{f}$, so that

$$u_i(\mathbf{r}) = \overline{V}_{ij}(\mathbf{r}, \mathbf{r}^*)f_j, \tag{4.1}$$

where, by definition, $\bar{V}_{ij}(\mathbf{r}, \mathbf{r}^*)$ is the required Green's function.

Defining $\bar{\Gamma}_{ij}$ as the Fourier transform of this tensor Green's function $\bar{V}_{ij}$, i.e.

$$\bar{\Gamma}_{ij} = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{V}_{ij} \exp\left[-i(k_1 r_1 + k_2 r_2)\right] dr_1\, dr_2, \tag{4.2}$$

and by taking the Fourier transform of (4.1), it is seen that

$$\Gamma_i = \bar{\Gamma}_{ij} f_j, \tag{4.3}$$

where Γ_i is the Fourier transform of u_i. Defining a quantity Γ_{ij} by the relation

$$\bar{\Gamma}_{ij} = \Gamma_{ij} \exp\left[-i(k_1 r_1^* + k_2 r_2^*)\right], \tag{4.4}$$

we can rewrite (4.3) as

$$\Gamma_i = \exp\left[-i(k_1 r_1^* + k_2 r_2^*)\right] \Gamma_{ij} f_j. \tag{4.5}$$

Substituting the value of Γ_1 obtained at the end of §3 into (4.5) gives the values of Γ_{11} and Γ_{13} by taking successively $f = (1, 0, 0)$ and $f = (0, 0, 1)$. Thus

$$\Gamma_{11} = (G_1 - ik_1 q^{-1} B_1 r_3) \exp\left[-qr_3\right] + \left[H_1 + ik_1 q^{-1} D_1 r_3\right] \exp\left[qr_3\right] + \tfrac{1}{16}\pi^{-2}\left[2 - k_1^2 q^{-2}(1 + q|r_3 - r_3^*|)\right] q^{-1} \exp\left[-q|r_3 - r_3^*|\right], \tag{4.6}$$

$$\Gamma_{13} = \left[G_3 - ik_1 q^{-1} B_3 r_3\right] \exp\left[-qr_3\right] + \left[H_3 + ik_1 q^{-1} D_3 r_3\right] \exp\left[qr_3\right] - \tfrac{1}{16}\pi^{-2} ik_1 q^{-1}(r_3 - r_3^*) \exp\left[-q|r_3 - r_3^*|\right]. \tag{4.7}$$

Similarly the values of Γ_{21}, Γ_{23}, Γ_{31} and Γ_{33} may be obtained as

$$\Gamma_{21} = \left[E_1 - ik_2 q^{-1} B_1 r_3\right] \exp\left[-qr_3\right] + \left[F_1 + ik_2 q^{-1} D_1 r_3\right] \exp\left[qr_3\right] - \tfrac{1}{16}\pi^{-2} k_1 k_2 q^{-3}(1 + q|r_3 - r_3^*|) \exp\left[-q|r_3 - r_3^*|\right], \tag{4.8}$$

$$\Gamma_{23} = \left[E_3 - ik_2 q^{-1} B_3 r_3\right] \exp\left[-qr_3\right] + \left[F_3 + ik_2 q^{-1} D_3 r_3\right] \exp\left[qr_3\right] - \tfrac{1}{16}\pi^{-2} ik_2 q^{-1}(r_3 - r_3^*) \exp\left[-q|r_3 - r_3^*|\right], \tag{4.9}$$

$$\Gamma_{31} = \left[A_1 + B_1 r_3\right] \exp\left[-qr_3\right] + \left[C_1 + D_1 r_3\right] \exp\left[qr_3\right] - \tfrac{1}{16}\pi^{-2} ik_1 q^{-1}(r_3 - r_3^*) \exp\left[-q|r_3 - r_3^*|\right], \tag{4.10}$$

$$\Gamma_{33} = \left[A_3 + B_3 r_3\right] \exp\left[-qr_3\right] + \left[C_3 + D_3 r_3\right] \exp\left[qr_3\right] + \tfrac{1}{16}\pi^{-2}(q^{-1} + |r_3 - r_3^*|) \exp\left[-q|r_3 - r_3^*|\right], \tag{4.11}$$

where, using the definitions (3.34)–(3.36) and (3.39) and letting $j = 1$ or 3, we have

$$A_j = S^{-1} \sum_{i=1}^{4} m_{ij} a_i, \quad B_j = S^{-1} \sum_{i=1}^{4} m_{ij} b_i,$$

$$C_j = -\tfrac{1}{16}\pi^{-2} m_{ij} - A_j, \quad D_j = S^{-1} \sum_{i=1}^{4} m_{ij} d_i, \tag{4.12}$$

$$E_j = -(n_{1j} e^q - n_{2j})/a_4, \quad F_j = (n_{1j} e^{-q} - n_{2j})/a_4, \quad G_j = -(n_{3j} e^q - n_{4j})/a_4,$$

$$H_j = (n_{3j} e^{-q} - n_{4j})/a_4, \tag{4.13}$$

$$m_{11} = ik_1 q^{-1} r_3^* \exp\left[-qr_3^*\right], \quad m_{21} = ik_1 (q^{-1} - r_3^*) \exp\left[-qr_3^*\right], \tag{4.14a}$$

$$m_{31} = -ik_1 q^{-1}(1 - r_3^*) \exp\left[-q(1 - r_3^*)\right], \quad m_{41} = ik_1 (q^{-1} - 1 + r_3^*) \exp\left[-q(1 - r_3^*)\right], \tag{4.14b}$$

$$m_{13} = (q^{-1} + r_3^*) \exp\left[-qr_3^*\right], \quad m_{23} = -r_3^* q \exp\left[-qr_3^*\right], \tag{4.14c}$$

$$m_{33} = (q^{-1}+1-r_3^*)\exp[-q(1-r_3^*)], \quad m_{43} = (1-r_3^*)q\exp[-q(1-r_3^*)], \tag{4.14d}$$

$$n_{11} = \tfrac{1}{16}\pi^{-2}k_1 k_2 q^{-3}(1+qr_3^*)\exp[-qr_3^*], \tag{4.15a}$$

$$n_{21} = ik_2 q^{-1}[B_1 e^{-q} - D_1 e^q] + \tfrac{1}{16}\pi^{-2}k_1 k_2 q^{-3}\{1+q(1-r_3^*)\}\exp[-q(1-r_3^*)], \tag{4.15b}$$

$$n_{13} = -\tfrac{1}{16}\pi^{-2}ik_2 q^{-1}r_3^*\exp[-qr_3^*], \tag{4.15c}$$

$$n_{23} = ik_2 q^{-1}[B_3 e^{-q} - D_3 e^q + \tfrac{1}{16}\pi^{-2}ik_2 q^{-1}(1-r_3^*)\exp[-q(1-r_3^*)], \tag{4.15d}$$

$$n_{31} = -\tfrac{1}{16}\pi^{-2}[2-k_1^2 q^{-2}(1+qr_3^*)]q^{-1}\exp[-qr_3^*], \tag{4.15e}$$

$$n_{41} = ik_1 q^{-1}[B_1 e^{-q} - D_1 e^q] - \tfrac{1}{16}\pi^{-2}[2-k_1^2 q^{-2}\{1+q(1-r_3^*)\}]q^{-1}\exp[-q(1-r_3^*)], \tag{4.15f}$$

$$n_{33} = k_1 n_{13}/k_2, \quad n_{43} = k_1 n_{23}/k_2. \tag{4.15g}$$

5. Calculation of migration velocity

The method of evaluation of the integrals involved in the equations (2.12), (2.13), (2.16) and (2.17)| for $h(\beta)$, $g(\beta)$, $f_1(\beta)$ and $f_2(\beta)$ is now discussed. The Fourier transform $\overline{\Gamma}_{ij}$ of the Green's function $\overline{V}_{ij}$ appearing in these integrals has been derived in §4. By taking the inverse transform of (4.2), one obtains the value of $\overline{V}_{ij}$ as

$$\overline{V}_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{\Gamma}_{ij}\exp[i(k_1 r_1 + k_2 r_2)]\,dk_1\,dk_2. \tag{5.1}$$

From this expression it is seen that

$$\frac{\partial\overline{V}_{ij}}{\partial r_1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ik_1\overline{\Gamma}_{ij}\exp i(k_1 r_1 + k_2 r_2)]\,dk_1\,dk_2. \tag{5.2}$$

Substituting from (4.4), (5.1) and (5.2) into (2.12) shows that the value of $h(\beta)$ is given, in terms of the Fourier transforms, by

$$h(\beta) = \int_{r_3=0}^{1}\int_{r_2=-\infty}^{\infty}\int_{r_1=-\infty}^{\infty}\Big\{\Big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ik_1\Gamma_{i1}\exp[i[k_1(r_1-r_1^*)+k_2(r_2-r_2^*)]] \times dk_1\,dk_2\Big]\Big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma_{i3}\exp[i[k_1(r_1-r_1^*)+k_2(r_2-r_2^*)]]\,dk_1\,dk_2\Big]\Big\}\,dr_1\,dr_2\,dr_3. \tag{5.3}$$

By the convolution theorem of Fourier transforms, this equation reduces to

$$h(\beta) = 4\pi^2\int_{r_3=0}^{1}\int_{k_2=-\infty}^{\infty}\int_{k_1=-\infty}^{\infty}[ik_1\Gamma_{i1}(k_1,k_2)\Gamma_{i3}(-k_1,-k_2)]\,dk_1\,dk_2\,dr_3. \tag{5.4}$$

In a similar way, it can be shown that

$$g(\beta) = 4\pi^2\int_{r_3=0}^{1}\int_{k_2=-\infty}^{\infty}\int_{k_1=-\infty}^{\infty}\Big\{(U_1(r_3)-U_1(\beta))]J_1 + \frac{\partial U_1(r_3)}{\partial r_3}J_2\Big\}\,dk_1\,dk_2\,dr_3, \tag{5.5}$$

$$f(\beta) = [\partial U_1(r_3)/\partial r_3]_{r_3=\beta}[f_1(\beta)+f_2(\beta)], \tag{5.6}$$

$$p(\beta) = [\partial U_1(r_3)/\partial r_3]_{r_3=\beta}[4f_1(\beta)+f_2(\beta)], \tag{5.7}$$

where

$$f_1(\beta) = 4\pi^2 \int_{r_3=0}^{1} \int_{k_2=-\infty}^{\infty} \int_{k_1=-\infty}^{\infty} \left\{ [U_1(r_3) - U_1(\beta)] J_3 + \frac{\partial U_1(r_3)}{\partial r_3} J_4 \right\} dk_1\, dk_2\, dr_3, \tag{5.8}$$

$$f_2(\beta) = 4\pi^2 \int_{r_3=0}^{1} \int_{k_2=-\infty}^{\infty} \int_{k_1=-\infty}^{\infty} \left\{ [U_1(r_3) - U_1(\beta)] J_5 + \frac{\partial U_1(r_3)}{\partial r_3} J_6 \right\} dk_1\, dk_2\, dr_3, \tag{5.9}$$

where Γ_{i1} and Γ_{i3} are given by (4.6)–(4.15) and

$$J_1 = ik_1 \Gamma_{i1}(k_1, k_2)\, \Gamma_{i3}(-k_1, -k_2), \quad J_2 = \Gamma_{31}(k_1, k_2)\, \Gamma_{13}(-k_1, -k_2), \tag{5.10a}$$

$$J_3 = ik_1 \Gamma_{i1, 3*}(k_1, k_2)\, \Gamma_{i3}(-k_1, -k_2), \quad J_4 = \Gamma_{31, 3*}(k_1, k_2)\, \Gamma_{13}(-k_1, -k_2), \tag{5.10b}$$

$$J_5 = k_1^2 \Gamma_{i3}(k_1, k_2)\, \Gamma_{i3}(-k_1, -k_2), \quad J_6 = -ik_1 \Gamma_{33}(k_1, k_2)\, \Gamma_{13}(-k_1, -k_2), \tag{5.10c}$$

and

$$\Gamma_{ij, 3*} = [\partial \Gamma_{ij} / \partial r_3^*]_{r_3^* = \beta}. \tag{5.11}$$

The triple integrations in (5.4)–(5.11) are performed by reducing them to double integrals by first changing to a triple integral over r_3, ρ and ϕ where

$$k_1 = \rho \sin \phi, \quad k_2 = \rho \cos \phi, \tag{5.12}$$

and then performing the ϕ integration analytically (see Vasseur 1973). The double integrals are then solved numerically on an IBM 360 computer, the results of these integrations being presented in the following sections.

6. Migration of a spherical particle in a stagnant fluid

In this section, the case of a spherical particle sedimenting in a fluid at rest (or undergoing a very slow motion), bounded by two plane walls W at $r_3' = 0$ and $r_3' = l$, is considered (see figure 3). If U_m is the characteristic velocity of the slow fluid motion between the walls, we require (as stated in §2) the condition

$$|V/U_m| \gg 1 \tag{6.1}$$

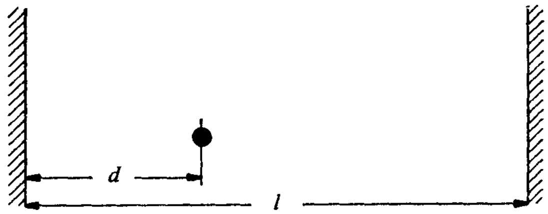
to be met. Making use of (2.8) and (5.4), it may be readily shown that the migration velocity v_l', normalized with respect to VRe, may be put in the form

$$\frac{v_l'}{VRe} = 24\pi^3 \int_{r_3=0}^{1} \int_{k_2=-\infty}^{\infty} \int_{k_1=-\infty}^{\infty} ik_1 \Gamma_{i1}(k_1, k_2)\, \Gamma_{i3}(-k_1, -k_2)\, dk_1\, dk_2\, dr_3, \tag{6.2}$$

where Γ_{i1} and Γ_{i3} are given by (4.6)–(4.15).

The results obtained from evaluating this triple integral in the manner described at the end of §5 are presented in figure 3. It is noted that the value of v_l'/VRe is positive for $0 < \beta < 0{\cdot}5$ and negative for $0{\cdot}5 < \beta < 1{\cdot}0$. This implies that the particle migrates away from the walls to an equilibrium position at $\beta = 0{\cdot}5$, i.e. at a position mid-way between the two plane walls.

The case of a spherical particle sedimenting in a stagnant fluid bounded by a single plane wall (at $r_3 = 0$) was studied by Cox & Hsu (1975), and it was found



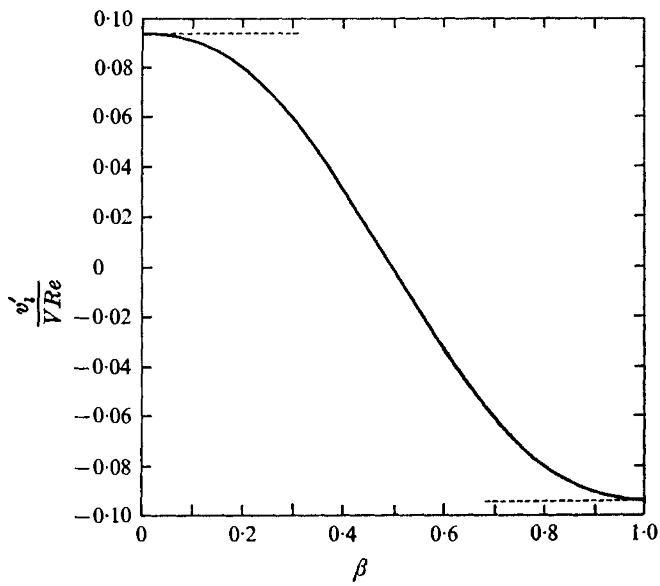


FIGURE 3. Lift velocity experienced by a spherical particle sedimenting in a stagnant fluid bounded by two plane walls: ——, present theory; - - - - - -, asymptotic value given by (6.3).

that the particle migrates away from the wall with a velocity v_l' given by

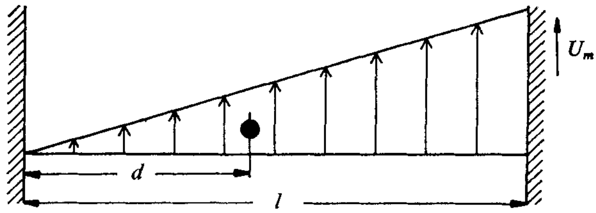
$$v_l'/VRe = \tfrac{3}{32} = 0{\cdot}09375. \tag{6.3}$$

This result (together with a similar one valid near $\beta = 1$) is plotted in figure 3 and it is seen that the results for the single wall and for the two walls become identical near the walls (i.e. as $\beta \to 0$ and $\beta \to 1$) and that in the neighbourhood of the walls, i.e. for $\beta < 0{\cdot}08$ and for $\beta > 0{\cdot}92$, the result (6.3) differs little from that predicted by the present theory. It has been observed experimentally that a spherical particle settling slowly near the wall of a circular tube (Oliver 1962; Karnis *et al.* 1966) or near the wall of a rectangular tube (Vasseur 1973) in an otherwise stagnant fluid migrates towards the tube axis as predicted by the present theory.

7. Migration of a spherical particle in a shear flow

7.1. *The flow field*

In this section, the migration velocity experienced by a spherical particle, neutrally buoyant or not, suspended in a simple shear flow is considered. The wall at



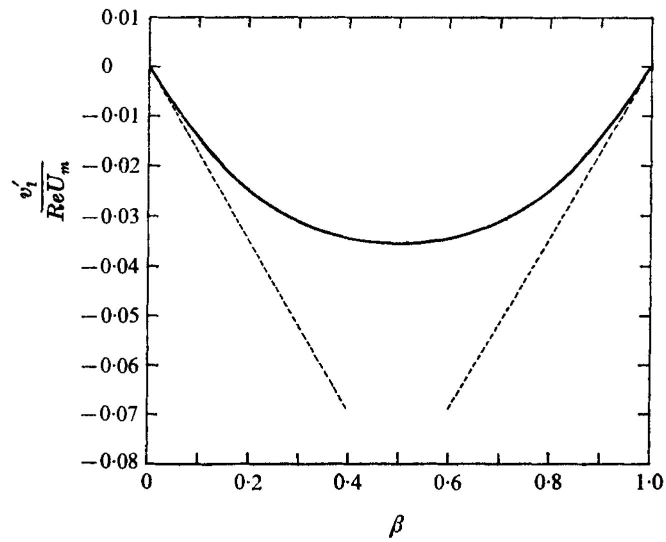


FIGURE 4. Lift velocity experienced by a non-neutrally buoyant spherical particle in a shear flow (sedimentation velocity V in the same direction as flow): ——, present theory; - - - - - -, asymptotic value given by (7.4).

$r_3' = 0$ is taken to be at rest with the fluid velocity upwards increasing linearly with r_3' and taking the value U_m, the characteristic velocity, at the wall $r_3' = l$ (see figure 4). Thus in terms of the dimensionless quantities defined by (2.7) the undisturbed shear flow is represented by

$$U_1(r_3) = r_3. \tag{7.1}$$

7.2. *A non-neutrally buoyant particle*

The case of a non-neutrally buoyant spherical particle suspended in the shear flow for which

$$\kappa^2 \ll |V/U_m| \ll 1 \tag{7.2}$$

is now considered. Substituting the value of $U_1(r_3)$ given in (7.1) into (5.5) one obtains the lift velocity v_l' (normalized by $Re\, U_m$) from (2.9) as

$$\frac{v_l'}{ReU_m} = -24\pi^3 \int_{r_3=0}^{1} \int_{k_2=-\infty}^{\infty} \int_{k_1=-\infty}^{\infty} \{(r_3-\beta) J_1 + J_2\} \, dk_1 \, dk_2 \, dr_3, \tag{7.3}$$

where J_1 and J_2 are defined in (5.10). This integral has been evaluated as described in §5 and the result is shown in figure 4. From (7.3), it is seen that the migration velocity changes sign when the direction of the shear flow is reversed. Thus, as shown in figure 4, when the particle sediments in the same direction as the stream velocity, migration is towards the stationary plane wall for all positions of the particle and, conversely, if the particle is sedimenting in the opposite direction to the stream velocity it migrates towards the moving wall.

The migration velocity experienced by a spherical particle suspended in a shear flow and located in the vicinity of a single plane wall (at $r_3 = 0$) was obtained by Cox & Hsu (1975) as

$$v_l'/Re\,U_m = -\tfrac{11}{64}\beta = -0{\cdot}171875\beta. \tag{7.4}$$

This result (together with one valid near $\beta = 1$ obtained by replacing β by $1-\beta$) is also plotted in figure 4 and it is seen that these results agree asymptotically with the present theory in the limits $\beta \to 0$ and $\beta \to 1$. It should be noted that the migration of a non-neutrally buoyant particle, as determined here by (2.9), is independent of whether the particle is free to rotate or not (see Cox & Brenner 1968).

7.3. *A neutrally buoyant particle*

Consider the case of a neutrally buoyant or almost neutrally buoyant particle for which

$$\kappa^2 \gg |V/U_m|. \tag{7.5}$$

By substituting for $U_1(r_3)$ from (7.1) into (5.6)–(5.9), one obtains from (2.10) the value of the lift velocity v_l' (normalized by $\kappa^2 a U_m^2/\nu$) experienced by a neutrally buoyant particle which is free to rotate and is suspended in the shear flow as

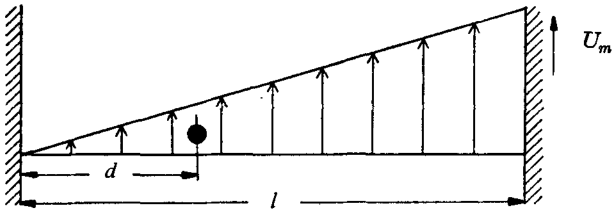
$$\frac{v_l'}{\kappa^2 a U_m^2/\nu} = \tfrac{40}{3}\pi^3 \int_{r_3=0}^{1}\int_{k_2=-\infty}^{\infty}\int_{k_1=-\infty}^{\infty} \{[(r_3-\beta)(J_3+J_5)]+(J_4+J_6)\}\,dk_1\,dk_2\,dr_3, \tag{7.6}$$

while similarly, from (2.11), the lift velocity v_l' experienced by a neutrally buoyant particle which is not allowed to rotate is

$$\frac{v_l'}{\kappa^2 a U_m^2/\nu} = \tfrac{16}{3}\pi^3 \int_{r_3=0}^{1}\int_{k_2=-\infty}^{\infty}\int_{k_1=-\infty}^{\infty} \{(r_3-\beta)(4J_3+J_5)+(4J_4+J_6)\}\,dk_1\,dk_2\,dr_3, \tag{7.7}$$

J_1, J_2, J_3 and J_4 being defined in (5.10). The results obtained by evaluating the integrals in (7.6) and (7.7) in the manner described in §5 are presented in figures 5 and 6 respectively. It is seen that a neutrally buoyant spherical particle suspended in a plane shear flow migrates away from the walls towards an equilibrium position at $\beta = 0{\cdot}5$, i.e. at a position mid-way between the two plane walls. It also appears that, in the vicinity of the walls, a particle which is prevented from rotating experiences a greater lift velocity than a particle which is free to rotate.

The case of a neutrally buoyant spherical particle, suspended in a shear flow, in the vicinity of a single plane wall (at $r_3 = 0$) was considered by Cox & Hsu (1975). They found that, if the particle is allowed to rotate, it migrates with a velocity



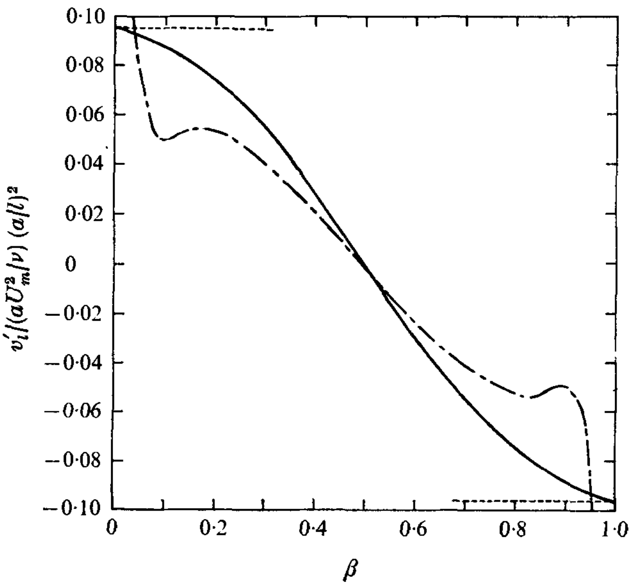


FIGURE 5. Lift velocity experienced by a neutrally buoyant spherical particle in a shear flow (particle freely rotates): ——, present theory; - - - - -, asymptotic value given by (7.8); —·—, values obtained by Ho & Leal (1974).

given by

$$v_l'/\kappa^2(aU_m^2/\nu) = \tfrac{55}{576} = 0{\cdot}095486, \tag{7.8}$$

whereas, if the particle is not allowed to rotate, it then migrates with a velocity given by

$$v_l'/\kappa^2(aU_m^2/\nu) = \tfrac{61}{576} = 0{\cdot}105903. \tag{7.9}$$

These results (together with one valid near $\beta = 1$) are also plotted in figures 5 and 6, and it is seen that they agree asymptotically with the present results in the limits of $\beta \to 0$ and $\beta \to 1$ and that in the neighbourhood of the walls, i.e. for $\beta < 0{\cdot}04$ and $\beta > 0{\cdot}96$, they differ very little from the present theory.

Ho & Leal (1974) have investigated theoretically the lateral migration of a freely rotating neutrally buoyant sphere in either shear flow or plane Poiseuille flow by a method similar to that used here except that, rather than using Fourier transforms, they evaluated the volume integrals (2.16) and (2.17) directly after having obtained an expression for $\overline{V}_{ij}$ as an integral. Their results have been plotted in figure 5 and it is seen that, while agreement is not too bad near the

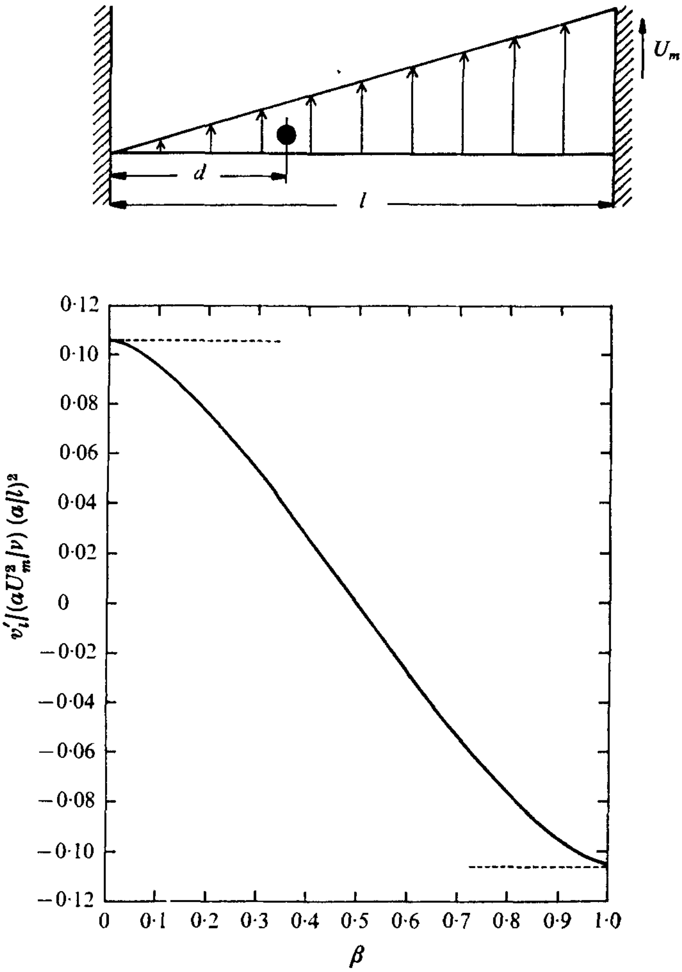
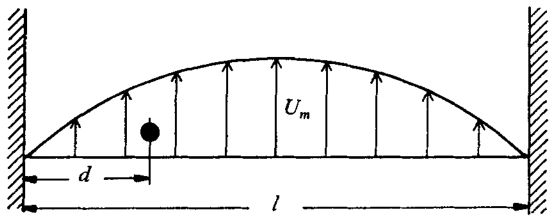


FIGURE 6. Lift velocity experienced by a neutrally buoyant spherical particle in a shear flow (particle is prevented from rotating): ——, present theory; - - - - -, asymptotic value given by (7.9).

centre $\beta = 0{\cdot}5$, it becomes poor near the walls and in fact their results do not seem to have there the asymptotic behaviour predicted by Cox & Hsu (1975). This may be due to poor convergence of the numerical computation when the sphere is close to a wall.

The migration of neutrally buoyant spherical particles in a Couette flow has been investigated experimentally by Halow (1968) and Halow & Wills (1970*a*, *b*), who found that the radial migration forces cause particles to assume an equilibrium position near the mid-point of the Couette system. This migration phenomenon was found to be dependent on the ratio of the diameter of the particle to the distance between the walls, and also on the Reynolds number based on the diameter of the particle. The present theory is qualitatively in agreement with these experimental results.



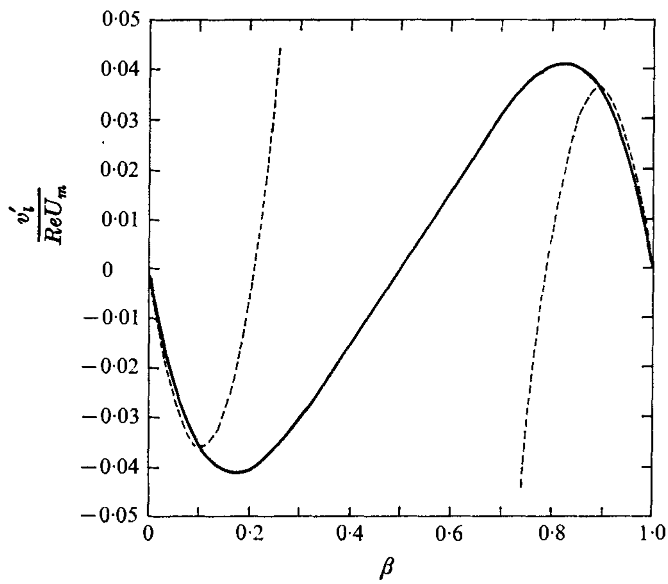


FIGURE 7. Lift velocity experienced by a non-neutrally buoyant spherical particle in a plane Poiseuille flow (sedimentation velocity V in the same direction as flow): ——, present theory; - - - - -, asymptotic value given by (8.3).

8. Migration of a spherical particle in a Poiseuille flow

8.1. *The flow field*

In this section, the migration velocity experienced by a spherical particle, neutrally buoyant or not, suspended in a two-dimensional Poiseuille flow between walls $r'_3 = 0$ and $r'_3 = l$ is considered (see figure 7). If the maximum flow velocity at $r'_3 = \frac{1}{2}l$ is taken as the characteristic velocity U_m, then the undisturbed Poiseuille flow may be expressed in terms of the dimensionless quantities defined by (2.7) as

$$U_1(r_3) = 4(r_3 - r_3^2), \tag{8.1}$$

giving a maximum velocity $U_1 = 1$ at the mid-point $r_3 = 0{\cdot}5$ and a zero velocity at the walls $r_3 = 0$ and $r_3 = 1$.

8.2. *A non-neutrally buoyant particle* ($\kappa^2 \ll |V/U_m| \ll 1$)

By substituting the value of $\dot{U}_1(r_3)$ as given by (8.1) into (5.5), one obtains a value for $g(\beta)$ which by making use of (2.9) gives the lift velocity v_l' (normalized by $Re\, U_m$) experienced by a non-neutrally buoyant spherical particle suspended in a two-dimensional Poiseuille flow as

$$\frac{v_l'}{Re\, U_m} = -96\pi^3 \int_{r_3=0}^{1} \int_{k_2=-\infty}^{\infty} \int_{k_1=-\infty}^{\infty} \{[(r_3-\beta)-(r_3^2-\beta^2)]J_1+(1-2r_3)J_2\} \times dk_1\, dk_2\, dr_3, \qquad (8.2)$$

J_1 and J_2 being defined by (5.10). This integral, evaluated as described in §5, gives the results presented in figure 7. It is seen that, for the case of a particle sedimenting with a velocity V in the same direction as that of the stream velocity U_m, the migration is away from the central position $\beta = 0{\cdot}5$ to the nearer of the two walls. However, if the particle is sedimenting in the direction opposite to that of the stream velocity, the reverse occurs and it migrates away from the walls to a position $\beta = 0{\cdot}5$ mid-way between the two plane walls.

For the case of a non-neutrally buoyant spherical particle suspended in a Poiseuille flow in the vicinity of a single plane wall (at $r_3 = 0$), it was found by Cox & Hsu (1975) that the migration velocity v_l' (normalized with $Re\, U_m$) experienced by the particle is given by

$$v_l'/Re\, U_m = -\tfrac{1}{32}[22-105\beta]\beta. \qquad (8.3)$$

This result (together with one valid near $\beta = 1$ obtained by replacing β by $1-\beta$) is also plotted in figure 7 and agrees asymptotically with the present theory in the limits $\beta \to 0$ and $\beta \to 1$, good agreement being obtained with the present theory when the particle is located in the neighbourhood of the walls, i.e. for $\beta < 0{\cdot}04$ and for $\beta > 0{\cdot}96$.

The behaviour of non-neutrally buoyant spherical particles suspended in laminar flow through a rectangular duct has been investigated experimentally by Repetti & Leonard (1964, 1966). By using a duct with cross-section of high aspect ratio they achieved an essentially two-dimensional Poiseuille flow, the particle migration being examined between the narrowly separated walls when it was confined to the mid-plane between the other walls. It was found that a spherical particle more dense than the fluid in a downflow travelled towards the nearer wall while a spherical particle less dense than the fluid in a downflow travelled towards the centre-plane; also, an increase in the migration rate was observed for increasing particle size and increasing flow rate, a dependence of this migration rate on the density difference between particle and fluid also being reported. All these experimental observations are qualitatively in agreement with the results of the present theory.

However, most of the experiments concerning the migration of non-neutrally buoyant spherical particles have been performed in tube flow (see, for instance, Eichhorn & Small 1964; Jeffrey & Pearson 1965; Denson *et al.* 1966), where an essentially similar migration phenomenon was observed, in that it was found that a spherical particle more dense than the fluid in an upflow (or a spherical

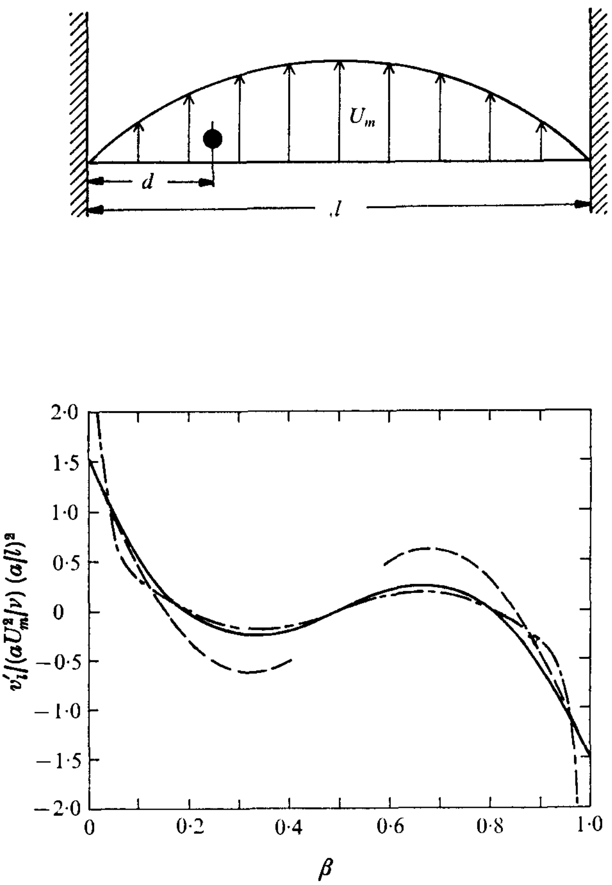


FIGURE 8. Lift velocity experienced by a neutrally buoyant spherical particle in a plane Poiseuille flow (particle freely rotates): ——, present theory; - - - - -, asymptotic value given by (8.8); — - —, values obtained by Ho & Leal (1974).

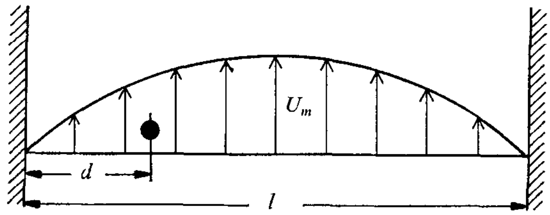
particle less dense in a downflow) migrates towards the tube axis, while a spherical particle less dense than the fluid in an upflow (or a spherical particle more dense in a downflow) migrates towards the tube wall.

8.3. *A neutrally buoyant particle* ($\kappa^2 \gg |V/U_m|$)

By substituting the value of $U_1(r_3)$ from (8.1) into (5.6)–(5.9), one obtains from (2.10) the value of the lift velocity v_l' (normalized by $(a/l)^2\, aU_m^2/\nu$) experienced by a neutrally buoyant spherical particle which is free to rotate and is suspended in a two-dimensional Poiseuille flow as

$$\frac{v_l'}{(a/l)^2\, aU_m^2/\nu} = \tfrac{640}{3}\pi^3(1-2\beta)\int_{r_3=0}^{1}\int_{k_2=-\infty}^{\infty}\int_{k_1=-\infty}^{\infty}\{[(r_3-\beta)-(r_3^2-\beta^2)]\,(J_3+J_5) + (1-2r_3)\,(J_4+J_6)\}\,dk_1\,dk_2\,dr_3, \qquad (8.4)$$

while similarly, from (2.11) one obtains the lift velocity experienced by a



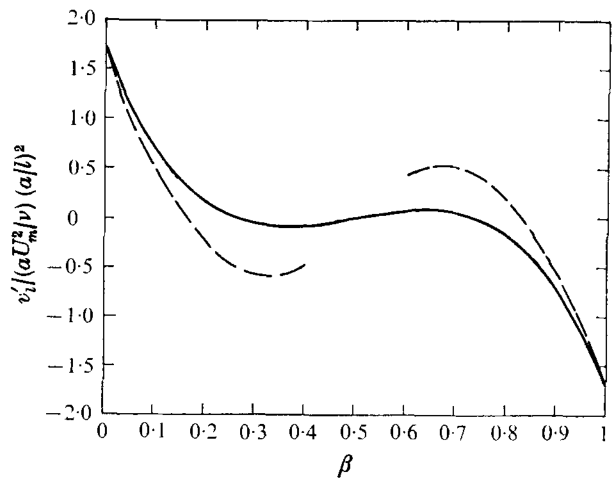


FIGURE 9. Lift velocity experienced by a neutrally buoyant spherical particle in a plane Poiseuille flow (particle is prevented from rotating): ——, present theory; - - - - -, asymptotic value given by (8.9).

neutrally buoyant spherical particle which is not allowed to rotate as

$$\frac{v_l'}{(a/l)^2\,aU_m^2/\nu} = \tfrac{256}{3}\pi^3(1-2\beta)\int_{r_3=0}^{1}\int_{k_2=-\infty}^{\infty}\int_{k_1=-\infty}^{\infty}\{[r_3-\beta)-(r_3^2-\beta^2)(4J_3+J_5) + (1-2r_3)(4J_4+J_6)\}\,dk_1\,dk_2\,dr_3, \tag{8.5}$$

J_3, J_4, J_5 and J_6 being defined in (5.10).

The integrals in (8.4) and (8.6) are presented in figures 8 and 9 respectively. It is seen that a neutrally buoyant particle suspended in a plane Poiseuille flow migrates away from both the walls and the centre-plane until it reaches a stable equilibrium position. For a particle that is free to rotate, this stable equilibrium position is at $\beta^* = 0{\cdot}19$ (and at $\beta^* = 0{\cdot}81$) while it is at $\beta^* = 0{\cdot}26$ (and $\beta^* = 0{\cdot}74$) for a particle that is prevented from rotating. For either case, the position $\beta^* = 0{\cdot}5$ is an unstable equilibrium position. The dependence of the position of stable equilibrium on particle rotation has been observed experimentally by Oliver (1962) for tube flow. He reported that rotating spherical particles reach an equilibrium position which is farther from the tube axis than that for non-rotating particles. The present theory is qualitatively in agreement with these observations.

For the case of a neutrally buoyant spherical particle suspended in a Poiseuille flow in the vicinity of a single plane wall, it was found by Cox & Hsu (1975) that for a particle which is allowed to rotate the lift velocity is given by

$$v_l'/(a/l)^2(aU_m^2/\nu) = \tfrac{5}{72}(1-2\beta)(22-146\beta), \tag{8.6}$$

while for a particle which is not allowed to rotate the lift velocity is given by

$$v_l'/(a/l)^2(aU_m^2/\nu) = \tfrac{1}{36}(1-2\beta)(61-368\beta). \tag{8.7}$$

These results are also plotted in figures 8 and 9 (together with results near $\beta = 1$ obtained by replacing β by $1-\beta$) and it is seen that they agree asymptotically with the present theory in the limits $\beta \to 0$ and $\beta \to 1$ (i.e. when the particle is located in the vicinity of the walls). The results obtained by Ho & Leal (1974) for a neutrally buoyant sphere free to rotate in a plane Poiseuille flow have also been plotted in figure 8 and agreement with the present theory is very good over the central portion ($0{\cdot}15 < \beta < 0{\cdot}85$), but, like their results for shear flow (see figure 5), agreement becomes poor when the particle is close to the walls. Furthermore, their results do not have the asymptotic behaviour near the walls predicted by Cox & Hsu (1975).

The migration of neutrally buoyant spherical particles flowing through a rectangular channel has been investigated experimentally by Yanizeski (1968). The channel was of high aspect ratio and the flow was essentially a two-dimensional Poiseuille flow. The effect of lateral migration was to move the particles away from both the mid-plane and the walls, with a particle taking up a stable position between the centre-plane and the walls. The smallest particles achieved equilibrium at $\beta^* = 0{\cdot}28$ while the largest particles travelled closer to the centre-plane. Such a dependence of the equilibrium position, for neutrally buoyant spherical particles, on particle size was also reported for tube flow by Karnis *et al.* (1966*a*). It was found that the largest particles, a/l near $0{\cdot}5$, travelled with their centres on the tube axis while progressively smaller particles travelled further from the axis. Since the particles used by Yanizeski were relatively large, it is not surprising that he obtained an equilibrium position $\beta^* = 0{\cdot}28$ different from the theoretical value $\beta^* = 0{\cdot}19$ predicted by the present theory.

More recently, the migration of neutrally buoyant spherical particles suspended in two- and three-dimensional Poiseuille flow has been studied experimentally by Tachibana (1973). The equilibrium positions reported in this investigation exhibited a great deal of scatter and, for reasons that are not clear from his paper, Tachibana presented sphere trajectories only for two cases with equilibrium positions of $\beta^* = 0{\cdot}20$ and $0{\cdot}80$. These equilibrium values are in good agreement with the theoretical values $\beta^* = 0{\cdot}19$ and $0{\cdot}81$ predicted by the present theory.

A large number of experimental investigations on the migration of neutrally buoyant spherical particles in tube flow have been conducted (see, for instance, Segré & Silberberg 1961, 1962; Oliver 1962; Karnis *et al.* 1966*a*, *b*; Jeffrey & Pearson 1965). A similar migration phenomenon was observed in that the neutrally buoyant particles migrate away from both the tube wall and the tube axis to an equilibrium position at $0{\cdot}5$ to $0{\cdot}6$ of the tube radius from the axis.

9. Migration of a spherical particle in a general flow

Consider a spherical particle suspended in a fluid contained between the vertical walls $r_3 = 0$ and $r_3 = 1$, the undisturbed fluid motion being in the vertical direction and varying with the r_3 co-ordinate. Since the undisturbed dimensionless fluid velocity $\mathbf{U}(\mathbf{r}) = (U_1, 0, 0)$ satisfies the Navier–Stokes equations, it is seen that it must be of the form

$$U_1 = ar_3^2 + br_3 + c, \tag{9.1}$$

where a, b and c are constants. The velocity of the walls at $r_3 = 0$ and $r_3 = 1$ would then be c and $a+b+c$ respectively.

Writing U_s ($\equiv r_3$) as the undisturbed shear flow discussed in §7 and U_p ($\equiv 4\,(r_3 - r_3^2)$) as the undisturbed Poiseuille flow discussed in §8, the general flow velocity given by (9.1) may be written

$$U_1 = -\frac{a}{4}U_p + (a+b)\,U_s + c. \tag{9.2}$$

Since $U_1 = c$ represents a uniform translation of the whole system and so cannot affect the particle migration, one may without loss of generality take U_1 to be a linear combination of the shear and Poiseuille flows. Thus we shall write

$$U_1 = pU_p + qU_s, \tag{9.3}$$

where p and q are constants.

For any particular situation under discussion we let the particle migration velocity for the shear flow U_s as calculated in § 7 be $(v_l')_s$ and for the Poiseuille flow U_p as calculated in §8 be $(v_l')_p$. Then, for a particle settling in a quiescent or nearly quiescent fluid for which $|V/U_m| \gg 1$, the migration velocity v_l' is independent of the flow (as is seen from (2.8) and (2.12)), its value being that determined in §6. However, for a non-neutrally buoyant particle for which $\kappa^2 \ll |V/U_m| \ll 1$, the migration velocity v_l' is seen from (2.9) and (2.13) to be linearly dependent upon U_1, so that upon substitution of U_1 from (9.3) one obtains

$$v_l' = p(v_l')_p + q(v_l')_s, \tag{9.4}$$

where $(v_l')_p$ and $(v_l')_s$ refer to the migration velocities for a non-neutrally buoyant particle and are derived respectively in §8.2 and §7.2.

In a similar manner, for a neutrally buoyant particle for which $|V/U_m| \ll \kappa^2$, the migration velocity v_l' must, from (2.10) and (2.14)–(2.17), be of the form

$$v_l' = \left[\frac{\partial}{\partial r_3}(pU_p + qU_s)\right]_{r_3 = r_3^*}\{Ap + Bq\}, \tag{9.5}$$

where A and B are functions of β only. Substitution of the values of U_p and U_s then yields

$$v_l' = \{4(1-2\beta)p + q\}\{Ap + Bq\}. \tag{9.6}$$

When $p = 1$ and $q = 0$, the undisturbed flow field is the Poiseuille flow U_p so that the corresponding migration velocity is $(v_l')_p$ (applicable to a neutrally buoyant particle). Thus

$$(v_l')_p = 4\,(1-2\beta)\,A. \tag{9.7}$$

Also, when $p = 0$ and $q = 1$, the undisturbed flow field is the shear flow U_s so that the corresponding migration velocity is $(v_l')_s$ (applicable to a neutrally buoyant particle). Thus

$$(v_l')_s = B. \tag{9.8}$$

Substitution of (9.7) and (9.8) into (9.6) yields for the migration velocity

$$v_l' = (v_l')_p p^2 + \left[\frac{(v_l')_p}{4(1-2\beta)} + 4(1-2\beta)(v_l')_s\right] pq + (v_l')_s q^2, \tag{9.9}$$

where $(v_l')_p$ and $(v_l')_s$ refer to the migration velocities of a neutrally buoyant particle and are derived respectively in §8.3 and §7.3. This result (9.9) is valid whether or not the particle is free to rotate (so long as v_l', $(v_l')_p$ and $(v_l')_s$ *all* refer either to a particle free to rotate or to a particle prevented from rotating).

10. Intermediate cases

The results obtained in the preceding sections may be summarized as follows:

(*a*) for a spherical particle sedimenting in a quiescent fluid or a very slowly moving one ($|V/U_m| \gg 1$) the migration velocity $(v_l')_a$ is given by

$$(v_l')_a \sim (aV^2/\nu)\, F_1(\beta), \tag{10.1}$$

(*b*) for a buoyant spherical particle ($\kappa^2 \ll |V/U_m| \ll 1$) the migration velocity $(v_l')_b$ is given by

$$(v_l')_b \sim -(aV/\nu)\, U_m F_2(\beta), \tag{10.2}$$

(*c*) for a neutrally buoyant spherical particle ($|V/U_m| \ll \kappa^2$) the migration velocity $(v_l')_c$ is given by

$$(v_l')_c \sim (aU_m^2/\nu)\, \kappa^2 F_3(\beta), \tag{10.3}$$

where the functions $F_1(\beta)$ and $F_2(\beta)$ are independent of whether the particle is free to rotate or not while $F_3(\beta)$ does depend on whether or not the particle is allowed to rotate. Cases intermediate between (*a*), (*b*) and (*c*) were also discussed by Cox & Brenner (1968). These are

(*d*) the case intermediate between (*a*) and (*b*) of a non-neutrally buoyant particle in a very slow flow for which $|V/U_m| \sim 1$. The migration velocity $(v_l')_d$ is then obtained as a linear combination of the results (2.8) and (2.9), namely

$$(v_l')_d = 6\pi(aV/\nu)\,[Vh(\beta) - U_m g(\beta)], \tag{10.4}$$

where $h(\beta)$ and $g(\beta)$ are defined by (2.12) and (2.13) respectively. This result is independent of whether the particle is free to rotate or not.

(*e*) the case intermediate between a neutrally buoyant (*c*) and a non-neutrally buoyant (*b*) particle for which $|V/U_m| \sim \kappa^2$. The migration velocity $(v_l')_e$ for such a particle when it is free to rotate is obtained as a linear combination of (2.9) and (2.10), namely

$$(v_l')_e = 6\pi \frac{aV}{\nu}\left[\tfrac{5}{9}\kappa^2 \frac{U_m^2}{V} f(\beta) - U_m g(\beta)\right], \tag{10.5}$$

while, if it is prevented from rotating, the migration velocity $(v_l')_e$ is obtained as a linear combination of (2.9) and (2.11), namely

$$(v_l')_e = 6\pi \frac{aV}{\nu}\left[\tfrac{2}{9}\kappa^2 \frac{U_m^2}{V} p(\beta) - U_m g(\beta)\right]. \tag{10.6}$$

where $f(\beta)$ and $p(\beta)$ are defined in (2.14)–(2.17).

Since case (d) is merely a composite of (a) and (b), while case (e) is a composite of (b) and (c), it follows that case (d) may be obtained by a simple superposition of the results obtained in §§ 6 and 7.2 for a spherical particle in a shear flow and the results obtained in §§ 6 and 8.2 for a spherical particle in a Poiseuille flow. Similarly, case (e) may be obtained by superposing the results of §§ 7.2 and 7.3 for a spherical particle in a shear flow and the results of §§ 8.2 and 8.3 for a spherical particle in a Poiseuille flow. In practical problems, one is unlikely to encounter particles which are precisely neutrally buoyant. For this reason, case (e) is likely to be of great interest in applications.

In terms of the functions $F_1(\beta)$, $F_2(\beta)$ and $F_3(\beta)$ appearing in (10.1)–(10.3), the migration velocities $(v_l')_d$ and $(v_l')_e$ for the intermediate cases (d) and (e) considered in this section may be rewritten as

$$(v_l')_d \sim \frac{aV}{\nu} V\left[F_1(\beta) - \frac{U_m}{V} F_2(\beta)\right], \tag{10.7}$$

and

$$(v_l')_e \sim \frac{aV}{\nu} U_m\left[\kappa^2 \frac{U_m}{V} F_3(\beta) - F_2(\beta)\right]. \tag{10.8}$$

For the case of a spherical particle suspended in a two-dimensional Poiseuille flow, it is found in §§ 6 and 8 that the functions $F_1(\beta)$, $F_2(\beta)$ and $F_3(\beta)$ have the following properties:

$$F_1(\beta)\begin{cases} > 0 & \text{for} \quad 0 \leqslant \beta < 0{\cdot}5, \\ = 0 & \text{for} \quad \beta = 0{\cdot}5, \end{cases} \tag{10.9}$$

$$F_2(\beta)\begin{cases} = 0 & \text{for} \quad \beta = 0, \\ > 0 & \text{for} \quad 0 < \beta < 0{\cdot}5, \\ = 0 & \text{for} \quad \beta = 0{\cdot}5, \end{cases} \tag{10.10}$$

$$F_3(\beta)\begin{cases} > 0 & \text{for} \quad 0 \leqslant \beta \leqslant \beta^*, \\ = 0 & \text{for} \quad \beta = \beta^*, \\ < 0 & \text{for} \quad \beta^* < \beta < 0{\cdot}5, \\ = 0 & \text{for} \quad \beta = 0{\cdot}5, \end{cases} \tag{10.11}$$

where β^* is the eccentric equilibrium position for a neutrally buoyant spherical particle.

As mentioned previously, the migration experienced by a spherical particle suspended in a two-dimensional Poiseuille flow has been investigated experimentally by Repetti & Leonard (1964, 1966). It was found that a more dense particle in a downflow migrated to the nearest wall while a less dense particle in a downflow migrated to the axis. It was found on the other hand that when the fluid–particle density differences were very small (of the order of $\pm 0{\cdot}05$ %), the

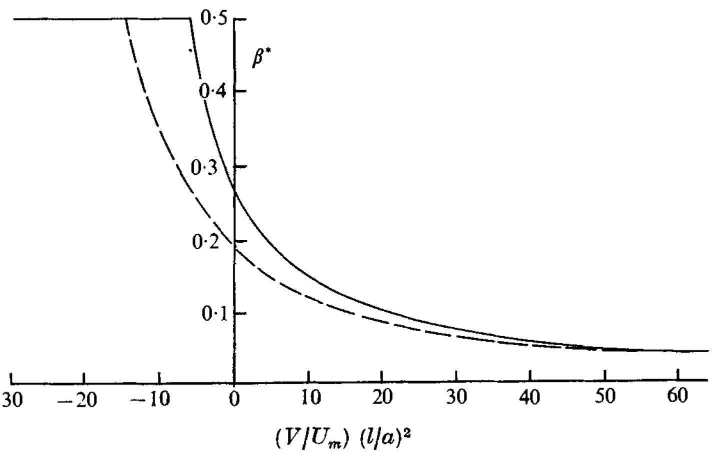


FIGURE 10. Equilibrium positions for the intermediate case between neutrally and non-neutrally buoyant: - - - -, a spherical particle freely rotating; ——, a spherical particle prevented from rotating.

migration of the particle ceased before it reached its extreme position at the wall or at the axis. Although the data of Repetti & Leonard are not sufficient to settle the matter unequivocally, it seems that the final stable equilibrium position will depend upon the density difference between the particle and the fluid.

The migration for the case intermediate between a neutrally and a non-neutrally buoyant particle suspended in a two-dimensional Poiseuille flow is described in the present theory by (10.8), according to which the stable equilibrium position β^* depends upon the dimensionless ratio $Ra = (U_m/V)\,(a/l)^2$, which may be either positive or negative. Using the results of §§ 8.2 and 8.3 for the functions $F_2(\beta)$ and $F_3(\beta)$, the equilibrium position β^* was calculated from (10.8), the results being presented in figure 10. For the case of a neutrally buoyant particle ($Ra = 0$), it is seen that a particle which is allowed to rotate reaches an equilibrium position at $\beta^* = 0{\cdot}19$ while, if the particle is slightly buoyant, its equilibrium position lies nearer to the wall than 0·19 when $Ra > 0$ and nearer to the axis than 0·19 when $Ra < 0$. When $Ra \leqslant -6$ the particle behaves as a non-buoyant particle and migrates until it reaches its extreme position at the axis, i.e. at $\beta^* = 0{\cdot}5$. This case thus corresponds to the case of a particle more dense than the fluid in an upflow or to the case of a particle less dense than the fluid in a downflow. It is of interest to note that, when $Ra > 0$, the equilibrium position of the particle moves asymptotically towards the wall as the value of Ra increases. This result indicates that, according to this theory, the particle never reaches the wall for any value of Ra. This is not in agreement with the experimental observations. However, the present theory is not expected to predict correctly the behaviour of a particle located in the vicinity of a plane wall since it has been assumed that $a/d \ll 1$, that is, the particle should be at least a few diameters away from the wall. The case of a slightly buoyant particle which is prevented from rotating is also depicted in figure 10 and the same general observations made for the case of a particle which is allowed to rotate are applicable. It is also interesting

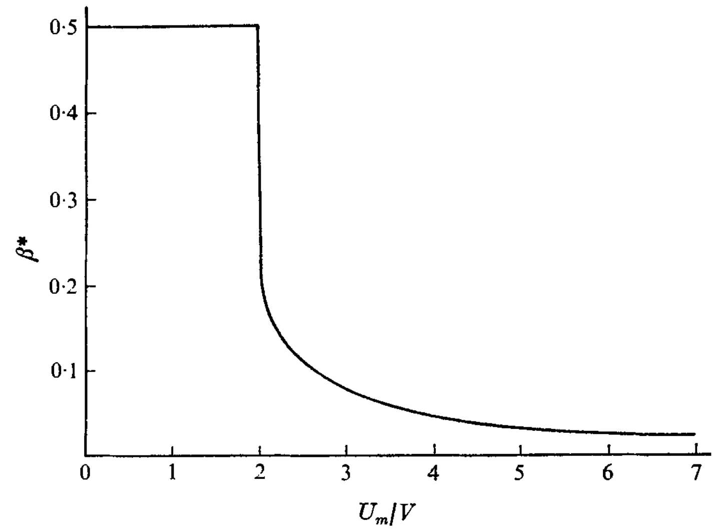


FIGURE 11. Equilibrium positions for the intermediate case where $U_m/V \sim 1$.

to note that the sedimenting velocity V is proportional to the factor $a^2\,(\Delta\rho)\,g/\mu$, where $\Delta\rho$ is the density difference between the particle and the fluid, and g is the local acceleration due to gravity. It follows that the equilibrium position β^* is independent of the particle size and depends only on the parameter $\mu U_m/\Delta\rho g l^2$. Thus particles of different density will accumulate on different streamlines, irrespective of particle size. As mentioned earlier, this phenomenon has been observed by Repetti & Leonard (1964, 1966) in a two-dimensional Poiseuille flow but has never been reported for the case of a particle suspended in a tube flow. However, the failure of previous investigators to observe this phenomenon in circular tubes is not surprising since, as pointed out by Brenner (1966), the range of density difference required to prevent migration of the particle all the way to the tube axis is very narrow.

According to (10.7), (10.9) and (10.10), it appears that there exist also eccentric stable equilibrium positions β^* for the intermediate case (*d*) of a non-neutrally buoyant particle in a very slow Poiseuille flow with $|V/U_m| \sim 1$. Using the results of §6 and §8.2, this equilibrium position β^* is plotted against the parameter U_m/V in figure 11. It is seen that the value of β^* changes very rapidly from 0·5 (the central position) to 0·2 as U_m/V is increased near the value of $U_m/V = 2$. This indicates that experimentally it would be very difficult to observe such an equilibrium position with β^* between 0·2 and 0·5 and in fact this phenomenon does not seem to have been observed in any previous experimental investigations. However, for values of β^* between 0 and 0·2, it should not be too difficult to observe experimentally this phenomenon since, for this range, the dependence on U_m/V is not so critical. Furthermore, it should be observed that, according to figure 11, the equilibrium position $\beta^* = 0$ is never reached, i.e. the particle will never migrate completely to the wall. This finding, however, is not expected to be valid because, as mentioned previously, the theory is probably incorrect when the particle is very close to the wall. The equilibrium position β^* occurs in the range

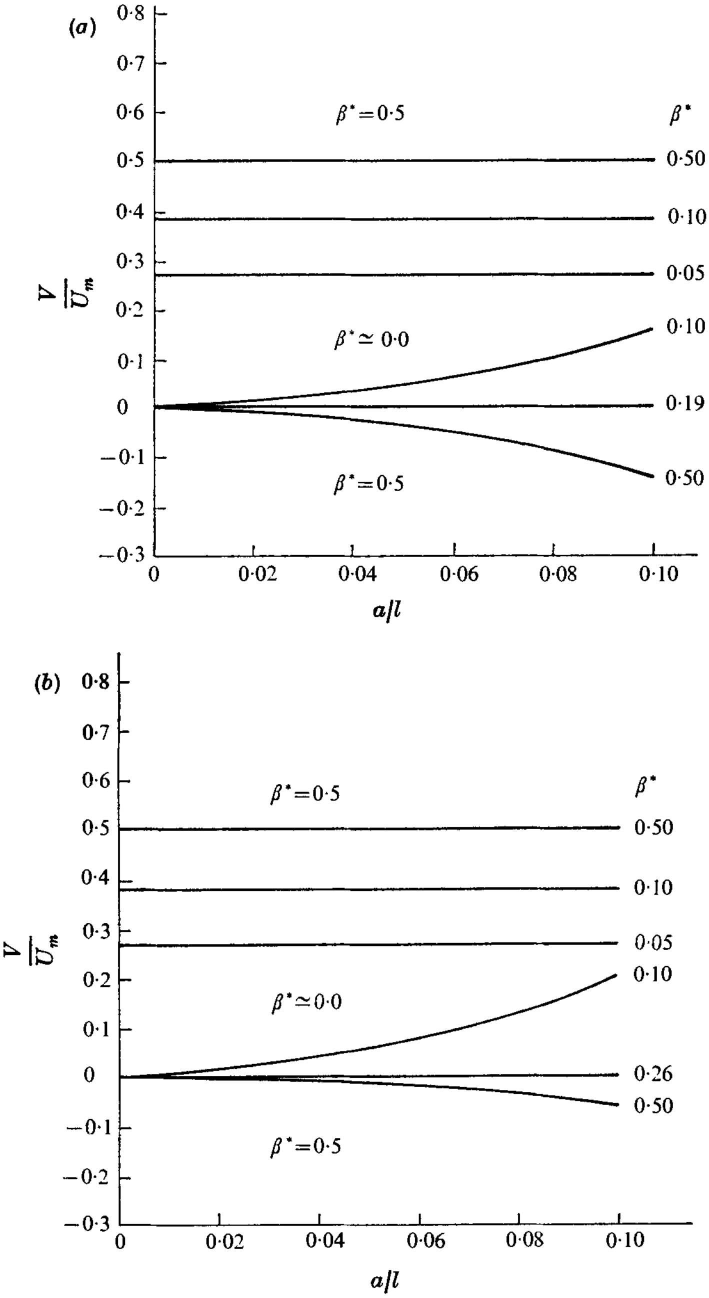


FIGURE 12. Equilibrium positions in terms of values of V/U_m and a/l for a spherical particle in a plane Poiseuille flow (a) for a particle allowed to rotate; and (b) for a particle prevented from rotating.

$0 < \beta^* < 0.5$ only if U_m/V is positive since for U_m/V negative migration is always to the central position $\beta^* = 0.5$ as is indicated by (10.7), (10.9) and (10.10). Thus the equilibrium position with $0 < \beta^* < 0.5$ will be observed only for the case of a particle less dense than the fluid in an upflow or for a particle more dense than the fluid in a downflow.

Thus a spherical particle suspended in a plane Poiseuille flow experiences an equilibrium position β^* which is in general a function of the ratio of the sedimenting velocity to the stream velocity V/U_m and of the particle to duct size ratio a/l. Using the results of figures 10 and 11, the values of the final equilibrium position β^* have been plotted on the a/l, V/U_m plane in figure 12(*a*) for the case of a particle which is allowed to rotate and in figure 12(*b*) for a particle prevented from rotating. It is seen that, for large positive values of V/U_m, the equilibrium position β^* of the particle is independent of the non-dimensional particle size a/l (at least to the order of approximation used in the theory) while, for smaller values of V/U_m, the equilibrium position depends upon a/l (except for $V/U_m = 0$, i.e. for the case of a neutrally buoyant particle).

11. Summary and conclusions

The lateral migration of a spherical particle suspended in a simple Couette flow and in a two-dimensional Poiseuille flow has been studied theoretically. The velocity v_l' of such a lateral migration has been found (under certain restrictions) in terms of the Reynolds number $Re = aU_m/\nu$ based on the undisturbed flow, the dimensionless particle size a/l, the ratio of the sedimentation velocity of the particle to the stream velocity V/U_m and the position of the particle in relation to the walls $\beta = d/l$. Furthermore, the following additional results were obtained: first, a non-neutrally buoyant particle suspended in a plane Poiseuille flow migrates towards the nearest wall if $U_m/V > 0$ and migrates away from the walls until it reaches an equilibrium position at a distance mid-way between the two walls if $U_m/V < 0$. Similarly, a particle suspended in a Couette flow (with one wall stationary) migrates towards the moving wall when $U_m/V < 0$ and towards the stationary wall when $U_m/V > 0$. On the other hand, a particle sedimenting in a stagnant fluid bounded by two vertical plane walls migrates until it reaches an equilibrium position at a distance mid-way between the two walls. These results are independent of whether the particle is free to rotate or not.

Second, a neutrally buoyant particle suspended in a plane Poiseuille flow migrates away from both the walls and the mid-plane until it reaches an eccentric equilibrium position β^*. For a particle which is free to rotate, this equilibrium position is at $\beta^* = 0.19$ (and $\beta^* = 0.81$) while, for a particle which is prevented from rotating, it is at $\beta^* = 0.26$ (and $\beta^* = 0.74$). Similarly, a particle suspended in a Couette flow is repelled by the walls and reaches an equilibrium position midway between the two plane walls.

Third, an almost neutrally buoyant particle suspended in a plane Poiseuille flow may experience equilibrium positions β^* at any distance between the walls and the centre-plane. Such equilibrium positions depend essentially on the density difference between the particle and the fluid and on whether the particle is allowed to rotate or not. Furthermore, the present theory predicts that there also exist eccentric equilibrium positions for the intermediate case of a non-neutrally buoyant particle in a slowly moving fluid. These equilibrium positions are independent of whether the particle is allowed to rotate or not and depend very critically on the ratio U_m/V and occur only if this ratio is larger than but not too much larger than $+2$.

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